ON THE CONSECUTIVE κ-OF-ν SYSTEM

by

CYRUS DERMAN, GERALD J. LIEBERMAN and SHELDON M. ROSS

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(SEE ABSTRACT)

ABSTRACT

We consider the consecutive k-of-n system in which there are n components linearly ordered. Each component either functions or fails and the system is said to be failed if any k consecutive components are failed. Let $r(\underline{p}) = r(p_1, \ldots, p_n)$ denote the probability that the system does not fail given that the components are independent, component i functions with probability p_i, i = 1, ..., n . The function $r(\underline{p})$ is called the reliability function.

We study the above system both when the components are linearly ordered and also when they are arranged in a circular order. In Section 2, we consider the case where all p_i are indentical and present a recursion for obtaining the reliability of a consecutive k-of-n in terms of the reliability of a consecutive k - 1 of n system. This yields simple explicit formulas when k is small and differs from the recursion obtained in [1]. In Section 3, we show how upper and lower bounds on $r(\underline{p})$ can be simply obtained. In Section 4, we consider a dynamic version in which each component independently functions for random time having distribution F . We show that when F is increasing failure rate (IFR), then system lifetime is also IFR only in the circular case when k = 2 . In Section 5, we consider a sequential optimization model in the linear k = 2 case. In this model, components are put in place one at a time with complete knowledge as to whether the previous component has worked or not. We show that the optimal policy is such that whenever a success occurs we follow it with the worst of the remaining components and whenever a failure occurs we follow it with the best of the remainder. In Section 6, we consider a nonsequential version of this. That is, the ordering must be fixed in advance.

ON THE CONSECUTIVE k-of-n SYSTEM

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1. INTRODUCTION

Chiang and Niu introduced (see [1]) the consecutive k-of-n system
in which there are n components linearly ordered. Each component
either functions or fails and the system is said to be failed if any
k consecutive components are failed. Let $r(\underline{p}) = r(p_1, \ldots, p_n)$
denote the probability that the system does not fail given that the
components are independent, component i functions with probability
p_i, $i = 1, \ldots, n$. The function $r(\underline{p})$ is called the reliability
function.

We study the above system both when the components are linearly
ordered and also when they are arranged in a circular order. In
Section 2, we consider the case where all p_i are identical and pre-
sent a recursion for obtaining the reliability of a consecutive k-of-n
in terms of the reliability of a consecutive $k - 1$ of n system.
This yields simple explicit formulas when k is small and differs from
the recursion obtained in [1]. In Section 3, we show how upper and
lower bounds on $r(\underline{p})$ can be simply obtained. In Section 4, we con-
sider a dynamic version in which each component independently functions
for a random time having distribution F. We show that when F is
increasing failure rate (IFR), then system lifetime is also IFR only in
the circular case when $k = 2$. In Section 5, we consider a sequential
optimization model in the linear $k = 2$ case. In this model, components

are put in place one at a time with complete knowledge as to whether the previous component has worked or not. We show that the optimal policy is such that whenever a success occurs we follow it with the worst of the remaining components and whenever a failure occurs we follow it with the best of the remainder. In Section 6, we consider a nonsequential version of this. That is, the ordering must be fixed in advance.

2. THE RELIABILITY FUNCTIONS WHEN $p_i \equiv p$

Before computing the reliability functions, we consider a combinatorial problem.

Suppose that j identical balls are to be placed in r distinct urns. Let $N_{j,r}(m)$ denote the number of ways this can be done subject to the requirement that at most m balls are placed in any one urns. When $m = 1$, this amounts to selecting j of the r urns, and so,

$$N_{j,r}(1) = \binom{r}{j} .$$

When $m = 2$, we can select i of the urns to contain 2 balls in $\binom{r}{j}$ ways and then we can select $j - 2i$ of the remaining $r - i$ urns to each contain 1 ball. Hence,

$$N_{j,r}(2) = \sum_i \binom{r}{i} N_{j-2i,r-i}(1) = \sum_i \binom{r}{i}\binom{r - i}{j - 2i} .$$

In general, by the same argument, we obtain the following recursion.

$$N_{j,r}(m) = \sum_i \binom{r}{i} N_{j-mi,r-i}(m - 1) .$$

Consider now the linear model when all the components have the same probability p of functioning. The number of ways that exactly j of the components can fail and the system function is equal to $N_{j,n-j+1}(k - 1)$ which can be seen by having the urns correspond to the places between successes and with one to the left of the first and one to the right of the last success. For instance if $k = 2$, $j = 3$, $n = 5$, then the 3 urns are

$$\sqcup \; s \; \sqcup \; s \; \sqcup \; .$$

As each arrangement of j failures and n - j successes has prob-ability $p^{n-j}(1 - p)^j$, we see that

Theorem 1:

In the linear case

$$r_L(p) = \sum_j N_{j,n-j+1}(k - 1)p^{n-j}(1 - p)^j .$$

Thus for instance,

$$(2.1) \qquad r_L(p) = \sum_j \binom{n - j + 1}{j} p^{n-j}(1 - p)^j \quad \text{when} \quad k = 2$$

$$(2.2) \quad r_L(p) = \sum_j \sum_i \binom{n - j + 1}{i}\binom{n - j + 1 - i}{j - 2i} p^{n-j}(1 - p)^j \quad \text{when} \quad k = 3 .$$

Remark:

The formula 2.1 was derived by Chiang and Niu in [1] by considering the number of ways that, of j failures and n - j successes, at least one success can be put between any two failures. This approach, however, does not seem to generalize beyond k = 2 as well as our approach which attempts to place the failures between the successes (as opposed to the reverse).

In the case of a circular system, choose any point on the circle between two components and let N and $\bar{N}$ denote respectively the number of failures observed until the first success, if we travel res-pectively in a clockwise or counterclockwise direction. Hence,

$$P\{N = j\} = P\{\bar{N} = j\} = p(1 - p)^{j-1} , \quad j = 0,1, \ldots, n .$$

Also, for $i < n - 1$, $P\{N + \bar{N} = i\}$ can be computed as if N and $\bar{N}$ were independent, and so

$$P\{N + \bar{N} = i\} = (i + 1)(1 - p)^i p^2 \ , \ i = 0, 1, \ldots, n - 2 \ .$$

Conditioning on $N + \bar{N}$ yields.

Theorem 2:

In the circular case

$$r_c(p) = p^2 \sum_{i=0}^{m-1} (i + 1)(1 - p)^i r_L(p \ , \ n - i - 2)$$

where $r_L(p, j)$ is the probability a j component linear system functions when each component functions with probability p .

Proof:

Given that $N + \bar{N} = i$, there is a run of i consecutive failures followed by successes on both sides. Hence, the remainder of the system acts as a $n - i - 2$ component linear system.$\|$

Remark:

Another way of obtaining the reliability is to let F_i denote the event that components $i, i+1, \ldots, i+k-1$ are all failed, $i = 1, \ldots, n - k + 1$. Then

$$1 - r(p) = P(\bigcup_i F_i)$$

$$= \sum P(F_i) - \sum_{i<j} \sum P(F_i F_j) + \ldots$$

and, for instance,

$$\sum P(F_i) = (n - k + 1)q^k \ , \ (q = 1 - p)$$

$$\sum_{i<j} \sum P(F_i F_j) = q^{k+1}(n - k) + (n - k - 1)q^{k+2} + (n - k - 2)q^{k+3}$$

$$+ \ldots + (n - 2k + 1)q^{2k}$$

and so on.

3. BOUNDS ON THE RELIABILITY FUNCTION

Chiang and Niu presented the minimal cut lower bound in [1]. Any k consecutive components constitute a minimal cut set (there are n - k + 1 in the linear case and n in the circular case). Letting E_i denote the event that at least one component of the ith minimal cut set functions i = 1, ..., r, then

$$r(\underline{p}) = P(E_1 E_2 \ldots E_r)$$

$$= P(E_1) P(E_2 \mid E_1) \ldots P(E_r \mid E_1, \ldots, E_{r-1}) .$$

The minimal cut lower bound uses the inequality

$$P(E_i \mid E_1, \ldots, E_{i-1}) \geq P(E_i) .$$

Thus, in the linear case, with $q_j = 1 - p_j$

$$r_L(\underline{p}) \geq \prod_i P(E_i) = \prod_{i=1}^{n-k+1} \left(1 - \prod_{j=i}^{i+k-1} q_j \right)$$

and, in the circular case

$$r_c(\underline{p}) \geq \prod_{i=1}^{n} \left(1 - \prod_{j=i}^{i+k-1} q_j \right)$$

where for $j > n$, $q_j \equiv q_{j-n}$.

We can improve these bounds by computing some of the joint probabilities. For example,

$$P(E_1 E_2) = 1 - P\left(E_1^c \cup E_2^c\right)$$

$$= 1 - \prod_{j=1}^{k} q_j - \prod_{j=2}^{k+1} q_j + \prod_{j=1}^{k+1} q_j$$

and, provided $k > 2$,

$$P(E_1 E_2 E_3) = 1 - P\left(E_1^c \cup E_2^c \cup E_3^c\right)$$

$$= 1 - \prod_1^{k} q_j - \prod_2^{k+1} q_j - \prod_3^{k+2} q_j$$

$$+ \prod_1^{k+1} q_j + \prod_2^{k+2} q_j .$$

Thus, for instance using this last identity, we can improve the bound on $r_L(p)$ to

$$r_L(p) \geq P(E_1 E_2 E_3) \prod_{i=4}^{n-k+1} \left(1 - \prod_{j=i}^{i+k-1} q_j\right).$$

An upper bound for $r(p)$ can be obtained by looking at minimal path sets--a minimal set of components whose functioning ensures that the system functions. However, as it is difficult to determine all the minimal path sets (in the linear case a vector of 1's and 0's will be a minimal path vector if any k consecutive elements contains at least one 1 and any $k + 1$ consecutive elements contains no more then two 1's) no effective upper bound was presented in [1]. To generate an upper bound, we first need the following lemma which was used for a special model in [2]. The proof presented in [2] differs from the one we give.

<u>Lemma 3</u>:

If N is a nonnegative random variable, then

$$P\{N > 0\} \geq E^2(N)/E(N^2) \ .$$

<u>Proof</u>:

$$E(N^2) = E(N^2 \mid N > 0)P(N > 0)$$

$$\geq (E(N \mid N > 0))^2 P(N > 0) \quad \text{by Cauchy-Schwarz}$$

$$= \frac{(E(N))^2}{P(N > 0)} \ . \mid\mid$$

To apply Lemma 3 let N denote the number of minimal cut sets that are down. That is,

$$(3.1) \qquad\qquad N = \sum_i I_i$$

where

$$I_j = \begin{cases} 1 & \text{if all components of the ith} \\ & \text{minimal cut set are failed} \\ 0 & \text{otherwise .} \end{cases}$$

As

$$r(\underline{p}) = 1 - P\{N > 0\} \ ,$$

we see that

$$r(\underline{p}) \leq 1 - \frac{E^2[N]}{E[N^2]} \ .$$

It is straightforward to compute $E[N]$.

$$E[N] = \begin{cases} \sum\limits_{i=1}^{n-k+1} \prod\limits_{j=i}^{i+k-1} q_j & \text{in the linear case} \\[2em] \sum\limits_{i=1}^{n} \prod\limits_{j=i}^{i+k-1} q_j & \text{in the circular case} \end{cases}$$

where again $q_{n+j} \equiv q_j$. As

$$E[N^2] = E\left[\sum I_i + \sum_{i \neq j}\sum I_i I_j\right]$$

$$= \sum E[I_i] + \sum_{i \neq j}\sum E[I_i I_j] \ ,$$

it is also straightforward, though messier, to compute $E[N^2]$. In the circular case when all the P_i are equal, the computations simplify. We first need the following lemma from [2].

<u>Lemma 4</u>:

Let N denote the number of events $A_1, \ldots, A_r$ that occur. If

$$P\{A_1 \mid N = j\} = \frac{j}{r} \quad \text{for all} \quad j \leq r \ ,$$

then

$$E[N \mid A_1] = E[N^2]/E[N] \ .$$

Proof:

$$E[N \mid A_1] = \sum jP\{N = j \mid A_1\}$$

$$= \sum jP\{A_1 \mid N = j\}P\{N = j\}/P(A_1)$$

$$= \sum j^2P\{N = j\}/rP(A_1)$$

$$= E[N^2]/rP(A_1) \ .$$

Also,

$$P(A_1) = \sum_j P(A_1 \mid N = j)P(N = j)$$

$$= E[N]/r$$

which completes the proof.$\|$

In the circular case if we let A_i , $i = 1, \ldots, n$ denote the
event that all the components of the ith minimal cut set are failed,
then it is clear that the conditions of Lemma 4 are satisfied when
$P_i \equiv p$. As

$$E[N \mid A_1] = 1 + 2[1 - p + (1 - p)^2 + \ldots + (1 - p)^{k-1}]$$

$$+ (n - 2k + 1)(1 - p)^k \ .$$

We see from Lemmas 3 and 4:

Theorem 5:

$$r_c(p) \le 1 - \frac{n(1 - p)^k}{1 + (n - 2k + 1)(1 - p)^k + 2\left(\frac{1 - p}{p}\right)[1 - (1 - p)^{k-1}]} \ .$$

4. A DYNAMIC VERSION

In this section, we suppose that all n components are initially working and continue to do so for a random time having distribution F . In addition, we suppose that the component lifetimes are independent and that F is an increasing failure rate (IFR) distribution. The latter statement means that

$$\lambda(t) = \frac{F'(t)}{1 - F(t)} \text{ is increasing in } t .$$

Let T denote the time at which the system fails. We are interested in determining when, if ever, T has an IFR distribution.

Theorem 6:

With k = 2 , in the circular case, T is IFR.

Proof:

Let N(t) denote the number of component failures at t . Then the failure rate function of T--call it λ_T--is given by

$$\lambda_T(t) = 2E[N(t) \mid T > t]\lambda(t)$$

where $\lambda(t)$ is the failure rate function of F .

However, Schechner has shown in [3] that in any structure in which all component lifetimes are independent and identically distributed

$$N(t) \mid T > t \text{ is stochastically increasing in } t .$$

Hence,

$$E[N(t) \mid T > t] \uparrow t$$

and the result follows. $\|$

There is no analog to Theorem 6 in the linear case. For instance, if $n = 3$, $k = 2$ and the component lives are exponential then if the system is still working at some moderate time t, it has a good chance of being in state $1 \textcircled{2} 3$ (where a circled number means that component has failed). However, if additional time passes and the system is still working, then it is probably either in $1 \textcircled{2} 3$ or in $\textcircled{1} 2 \textcircled{3}$. As $\textcircled{1} 2 \textcircled{3}$ is the better state as far as having a longer additional life, it follows intuitively (and, of course, can be numerically checked) that T is not IFR. A counterexample to T being IFR in the circular case is obtained when $n = 6$, $k = 3$. When four components are down and the system is still working, the state is of the form

$$\textcircled{6} \overset{1}{} \textcircled{2}$$
$$\textcircled{5} _4 \textcircled{3}$$

where a circled component means that it is failed. As the above state is at least as good (as far as additional life time) as any state where only three are failed and the system is working and strictly better, than those of the form

$$_6 \textcircled{1} {}_2$$
$$\textcircled{5} {}_4 \textcircled{3}$$

it follows that the system is better off if it is working and four are down than it is when three are down. Thus, again, T will not be IFR.

5. A SEQUENTIAL OPTIMIZATION PROBLEM

Suppose that we have n components with reliabilities $P_1, \ldots, P_n$.
The components are to be put into use one at a time. When put in use, we
immediately discover whether or not they work and can thus use this informa-
tion in deciding upon the next component. We say that the system fails if
we ever get two component failures in succession. Thus, we have a sequen-
tial version of the linear model with k = 2 . We are interested in determin-
ing the dynamic ordering of components so as to minimize the probability
of a failed system.

Theorem 7:

Suppose $P_1 \leq P_2 \leq \ldots \leq P_n$. The optimal strategy is to first put
component 1 into use. Afterwards, if the most recent component in use has
failed, then the next one should be the one with highest reliability; if
the most recent one has succeeded, then the next one should be the one with
lowest reliability.

Proof:

Consider any policy which doesn't follow the advice of the alleged
optimal policy—call it policy π and suppose first that it differs at
some time when the last component has failed. Specifically suppose that
a component has just failed and among the remaining components are i
and j where $P_i < P_j$ and suppose π calls for installing component
i . We show that π cannot be optimal by considering a new policy π'
which calls for first putting component j in, then acting as π would
if component i was put in, except that at the time that π would put

j in π' puts i instead. Now if i and j both work or both fail or if the components installed by π between i and j contain two consecutive failures or end with a failure, then π and π' are identical. So suppose that of i and j exactly one fails and one works and that the ones between i and j have no two consecutive failures and ends in a success. Now if i fails and j succeeds, then

$$\text{under } \pi : \text{ state = "failed"} \qquad \text{under } \pi' : \text{ state} = \alpha$$

whereas, if i succeeds and j fails, then

$$\text{under } \pi : \text{ state} = \alpha \qquad \text{under } \pi' : \text{ state = "failed"}$$

where α is the state that we have just had a failure and all components except i , j and those put in by π between i and j remain. As α is clearly a better state than "failed" and as the conditional probability of i failing and j succeeding given exactly one failing is higher than the reverse probability it follows that π' is better than π .

The above shows that it is optimal to follow a failed component with the best of the remaining ones.

It remains to prove that it is optimal to follow a success with the lowest reliability component--or equivalently, that the very first component used should be the one with lowest reliability. We will prove this by induction on n . As it is obvious for $n = 1 , 2 , 3$, assume for $n - 1$, and consider an n component problem. Suppose i is initially put in when $P_i > P_1 = \min_j P_j$ and suppose that an optimal policy is followed thereafter. Now if i succeeds, then the next component will be 1 by the induction hypothesis, if i fails, then the next component will be the best (by the first part of the proof) and

then it will be followed by 1 (by the induction hypothesis, since if in
this case the best one fails, it is irrelevant which component follows it).
Hence, the policy--call it π--is such that

$$i \ \text{success} \Rightarrow i \ , \ 1$$

$$i \ \text{failure} \Rightarrow i \ , \ n \ , \ 1$$

where $P_n = \max P_j$. Also note that if i is a success and 1 a failure
then by the first part of the proof π will then use component n .

Let us compare this with the policy--say π'--that initially puts 1
in and then

$$1 \ \text{success} \Rightarrow 1 \ , \ i$$

$$1 \ \text{failure} \Rightarrow 1 \ , \ n \ , \ i$$

and then continues optimally.

If 1 and i are both successes or both failures, then the two
policies do equally well. Otherwise:

if i is a success and 1 a failure,

under π : success, failure, n

under π' : failure, n , success,

if i is a failure and 1 a success,

under π : failure, n , success

under π' : success, failure, n .

Hence, in all the above cases, n must be a success or else the system fails. When n is a success, both policies do identically well.

Thus, it is also optimal to start with component 1 and the proof is complete.

6. THE NON-SEQUENTIAL OPTIMIZATION PROBLEM, $k = 2$

As in Section 5 suppose we have n components with reliabilities $p_1, p_2, \ldots, p_n$ with $p_1 \leq p_2 \leq \cdots \leq p_n$. The system is to be constructed at one time so as to maximize its reliability. That is, if we let ψ denote a permutation of the first n positive integers, the problem is to find a permutation ψ^* such that

$$r(\psi^*) = \max_{\psi} r(\psi) \; ,$$

where $r(\psi)$ is the reliability of the system arrangement $\psi(1), \psi(2), \ldots, \psi(n)$.

Let $q_i = 1 - p_i$, $i = 1, \ldots, n$. Thus, $q_1 \geq q_2 \geq \cdots \geq q_n$. For the case $n = 2$,

$$r(\psi) = 1 - q_{\psi(1)} q_{\psi(2)} \; ,$$

and either of the two possible arrangements is optimal. For the case $n = 3$,

$$r(\psi) = 1 - q_{\psi(1)} q_{\psi(2)} - q_{\psi(2)} q_{\psi(3)} + q_{\psi(1)} q_{\psi(2)} q_{\psi(3)} \; .$$

A permutation ψ that minimizes

$$q_{\psi(1)} q_{\psi(2)} + q_{\psi(2)} q_{\psi(3)}$$

maximizes $r(\psi)$. We may verify that $\psi^*(1) = 1$, $\psi^*(2) = 3$, $\psi^*(3) = 2$ is such a permutation. For the case $n = 4$, after some elementary algebra,

$$r(\psi) = 1 - [q_{\psi(1)} q_{\psi(2)} + q_{\psi(3)} q_{\psi(4)} + q_{\psi(2)} q_{\psi(3)} (1 - q_{\psi(1)} - q_{\psi(4)})] \; .$$

The permutation $\psi^*(1) = 1$, $\psi^*(2) = 4$, $\psi^*(3) = 3$, $\psi^*(4) = 2$ min-imizes the sum of the first two terms within the brackets while min-imizing, at the same time, the last term. Thus ψ^* maximizes $r(\psi)$.

Now, for any $n \geq 1$, we define $\tilde{\psi}$ according to the following scheme: $\tilde{\psi}(1) = 1$, $\tilde{\psi}(n) = 2$, $\tilde{\psi}(2) = n$, $\tilde{\psi}(n-1) = n-1$, $\tilde{\psi}(3) = 3$, $\tilde{\psi}(n-2) = 4$, $\tilde{\psi}(4) = n-2$, $\tilde{\psi}(n-3) = n-3$, ... etc. For $n = 2$, 3 , 4 , $\psi^* = \tilde{\psi}$. We conjecture, but have been unable to prove, that $\psi^* = \tilde{\psi}$ is an optimal permutation for all $n \geq 2$.

We turn our attention to the number of minimal cuts sets that fail, N , and consider

$$S(\psi) = E_\psi N$$

$$= \sum_{i=1}^{n} q_{\psi(i)} q_{\psi(i+1)} .$$

$\tilde{\psi}$ minimizes $S(\psi)$; that it does is a consequence of the following.

Proposition:

Suppose $n \geq 4$ and ψ satisfies one of the following conditions:

(i) $\psi(1) = n$, $\psi(n) = n - 1$, either $\psi(2) \neq 1$ or $\psi(n-1) \neq 2$

(ii) $\psi(1) = 1$, $\psi(n) = 2$, either $\psi(2) \neq n$ or $\psi(n-1) \neq n-1$,

then, in each case, there exists a permutation ψ' where $\psi'(1) = \psi(1)$, $\psi'(n) = \psi(n)$ and in case (i) $\psi'(2) = 1$, $\psi'(n-1) = 2$ or in case (ii) $\psi'(2) = n$, $\psi'(n-1) = n-1$ such that $S(\psi') \leq S(\psi)$.

Proof of (i):

Suppose $\psi(2) \neq 1$. Let k be such that $\psi(k) = 1$. Note that

$1 < k < n$. Define $\bar{\psi}$ by

$$\bar{\psi}(j) = \psi(k - j + 2) \ , \ j = 2, \ldots, k$$

$$\bar{\psi}(j) = \psi(j) \ , \ \text{otherwise.}$$

Then

$$S(\psi) - S(\bar{\psi}) = q_n(q_{\psi(2)} - q_1) + q_{\psi(k+1)}(q_1 - q_{\psi(2)})$$

$$= (q_n - q_{\psi(k+1)})(q_{\psi(2)} - q_1)$$

$$\geq 0 \ .$$

Now, if $\bar{\psi}(n - 1) = 2$, let $\psi' = \bar{\psi}$. If $\bar{\psi}(n - 1) \neq 2$ then there is a k , $2 < k < n$, such that $\bar{\psi}(k) = 2$. Define ψ' by

$$\psi'(j) = \bar{\psi}(n + k - j - 1) \ , \ j = k, \ldots, n - 1$$

$$\psi'(j) = \bar{\psi}(j) \ , \ \text{otherwise.}$$

Then

$$S(\bar{\psi}) - S(\psi') = q_{\bar{\psi}(k-1)}(q_2 - q_{\bar{\psi}(n-1)}) + q_{n-1}(q_{\bar{\psi}(n-1)} - q_2)$$

$$= (q_{\bar{\psi}(k-1)} - q_{n-1})(q_2 - q_{\bar{\psi}(n-1)})$$

$$\geq 0 \ .$$

Therefore

$$S(\psi) - S(\psi') = S(\psi) - S(\bar{\psi}) + S(\bar{\psi}) - S(\psi')$$

$$\geq 0 \ .$$

If $\psi(2) = 1$, define ψ' as above using $\bar{\psi} = \psi$.

The proof of (ii) follows from a similar construction.

Without changing the problem we can regard the system whose expected number of minimal cuts is to be minimized as consisting of $n + 2$ components with $p_{n+1} = p_{n+2} = 1$; the two additional perfectly reliable components have assigned positions at the extremes. From this point of view, upon repeated application of the proposition, initially invoking condition (i), it follows that $\tilde{\psi}$ minimizes $S(\psi)$.

REFERENCES

[1] Chiang, D. and S. C. Niu, "Reliability of Consecutive k out-of-n,"
     to appear in <u>IEEE Transactions on Reliability</u>.

[2] Matula, D., "The Largest Clique Size in a Random Graph," Technical
     Report CS 7608, Department of Computer Science and Engineering,
     Southern Methodist University, Dallas, Texas, (April 1976).

[3] Schechner, Z., "Conditionally Increasing Processes," ORC 80-24,
     Operations Research Center, University of California, Berkeley,
     (November 1980).